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# Nonlinear Ekman–Hartmann layers and the flow outside

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We consider in this paper the hydromagnetic flow of a viscous, incompressible, electrically conducting fluid induced by the differential rotation of an insulating boundary. A uniform magnetic field parallel to the axis of rotation is applied at a large distance from the interface. The method of matched asymptotic expansions is employed to study the hydromagnetic coupling between different regions of fluid flow and the interior of the insulating boundary.

## 1. Introduction

In this paper we study the influence of differential rotation of the container on the behaviour of a rotating, incompressible, electrically conducting fluid under hydromagnetic interaction. It has been established by Benton & Loper (1969) and Chawla (1972) that the evolution (spin-up) of such a flow from rigid-body rotation leads to the formation of a double-decker boundary layer on the parts of the container in contact with the fluid. Our aim here is to analyse the flow and the magnetic field in both the regions in their nonlinear forms. The specific purpose of the present paper is to study the intensity of hydromagnetic coupling between different regions of fluid flow and to investigate the nonlinear changes in the applied magnetic field which may be supported by the (inner) boundary layer on the rotating boundary. Surprisingly, the dynamics of the outer region are crucial for both of these physical phenomena.

Some aspects of the physical problem described here have been considered by Gilman & Benton (1968) and Benton & Chow (1972). Their analyses, however, preclude the discussion of the outer magnetic diffusion region on the assumption that it is spatially uniform. Loper (1972), on the other hand, has studied a nonlinear resistive boundary layer which is supported by a steady forcing introduced through axial electric currents in the far field. He excluded the viscous and nonlinear inertia forces from his analysis. Moreover, the existence of Loper's boundary layer depends upon the axial field, axial current and rotation being of the same sign. The present paper differs in a fundamental way from that of Loper (1972) in that it includes the effects of the viscosity of the fluid. We find that viscosity, in combination with other effective forces, manifests itself through its ability to provide a proper balance in whole of the flow regime affected by hydromagnetic interaction.

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The study of the magnetohydrodynamics of rotating fluids has applications in several situations of geophysical and astrophysical interest. It is likely to explain the observed phenomenon of secular variations and maintenance of the geomagnetic field. It may also establish the degree of electromagnetic coupling of the electric currents flowing in the convective envelope of some of the stars to their interior. The study of hydromagnetic spin-up is somewhat connected with the temporal evolution of rotating magnetic stars. A direct application of the results derived here to any of these physical problems is, however, severely hampered by the idealization employed in the paper.

## 2. The mathematical formulation

We consider the situation in which a homogeneous fluid of constant viscosity $\nu$ and magnetic diffusivity $\eta$ is bounded by a rigid insulating half-space $Z \leqslant 0$ with the whole space permeated by a magnetic field of constant strength $H_0$ in the $Z$ direction. This system is in a state of rigid-body rotation of angular velocity $\Omega$ about the $Z$ axis. The insulator, in contact with the fluid, is now assumed to be rotating with angular velocity $\Omega(1+\epsilon)$ whereas the fluid at infinity is left in the undisturbed state of rigid-body rotation. The magnetic field in the fluid at infinity is constrained to be in the axial direction, and all the boundary conditions are to be written accordingly.

We take cylindrical polar co-ordinates $(r, \theta, Z)$ with an accompanying fluid velocity $\mathbf{V}$, magnetic field $\mathbf{H}$ and hydromagnetic pressure $p$. For consistency with the axial symmetry and the continuity equations, we define

$$\mathbf{V} = r\Omega[F'\hat{\mathbf{r}} + (1+G)\hat{\boldsymbol{\theta}}] - 2(\nu\Omega)^{\frac{1}{2}} F\hat{\mathbf{z}}, \tag{2.1}$$

$$\mathbf{H} = -(H_0 r/\eta)(\nu\Omega)^{\frac{1}{2}}[N'\hat{\mathbf{r}} + M\hat{\boldsymbol{\theta}}] + \mathrm{H}_0\left(1+\frac{2\nu}{\eta}N\right)\hat{\mathbf{z}}, \tag{2.2}$$

$$p = \tfrac{1}{2}\rho r^2\Omega^2 + P, \tag{2.3}$$

where $F$, $G$, $M$, $N$ and $P$ are functions of the dimensionless variable $z = (\Omega/\nu)^{\frac{1}{2}}Z$ and a prime denotes differentiation with respect to $z$. Moreover, $\rho$ is the density of the fluid and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ are unit vectors in the $r$, $\theta$ and $z$ directions respectively. Substituting (2.1)–(2.3) in the basic hydromagnetic equations leads to

$$F''' + 2G - 2\lambda N'' = F'^2 - 2FF'' - G^2 - 2\lambda\sigma[N'^2 - 2NN'' - M^2], \tag{2.4}$$

$$G'' - 2F' - 2\lambda M' = 2(GF' - FG') - 4\lambda\sigma(MN' - NM'), \tag{2.5}$$

$$M'' - G' = 2\sigma(NG' - FM'), \tag{2.6}$$

$$N'' - F' = 2\sigma(NF' - FN'), \tag{2.7}$$

where $\lambda$ and $\sigma$ are the dimensionless parameters defined by

$$\lambda = \mu H_0^2/2\rho\eta\Omega, \quad \sigma = \nu/\eta. \tag{2.8}$$

Here $\sigma$ is the magnetic Prandtl number and $\lambda$ measures the strength of the magnetic force relative to the centrifugal force. In terms of $M$ and $N$, the current density vector is

$$\mathbf{J} = (H_0/\eta)[r\Omega M'\hat{\mathbf{r}} - r\Omega N''\hat{\boldsymbol{\theta}} - 2(\nu\Omega)^{\frac{1}{2}} M\hat{\mathbf{z}}]. \tag{2.9}$$

Because no currents flow in the insulating half-space, clearly $M$ must be zero at $z = 0$. A bounded solution of (2.7) within the insulator is $N' = 0$ (for $z \leqslant 0$). Also, all perturbations to the applied magnetic field must vanish at large distances from the interface $z = 0$. Thus the appropriate boundary conditions on the velocity and magnetic fields are

$$\left.\begin{array}{l} F'(0) = 0, \quad G(0) = \epsilon, \quad F(0) = 0, \quad F'(\infty) = 0 = G(\infty), \\ M(0) = 0, \quad N'(0) = 0, \quad M(\infty) = 0 = N(\infty). \end{array}\right\} \tag{2.10}$$

It needs to be pointed out that these boundary conditions are different from those imposed by Gilman & Benton (1968), Benton & Chow (1972) and Loper (1972). In the present notation Gilman & Benton and Benton & Chow use the following conditions on the magnetic field:

$$M(0) = N''(0) = 0, \quad N'(0) = \text{constant}, \quad N(\infty) = N'(\infty) = M'(\infty) = 0.$$

The weakness of these conditions is exposed immediately as the continuity of the tangential component $N'$ leads to $N = N'(0)z + \text{constant}$ for $z \leqslant 0$; this solution blows up as $z \to -\infty$. In order to maintain a steady forcing by electromagnetic forces, Loper (1972) introduces a non-zero axial current in the far field and uses the following conditions at the fluid–solid interface:

$$\left.\begin{array}{c} F(0) = 0, \quad M(0) = 0, \quad N'(0) = 0, \quad M'(0) = \text{constant (prescribed)}, \\ N(0) = \text{constant (prescribed)}. \end{array}\right\} \tag{2.11}$$

As pointed out by Loper (1972), these conditions need to be revised when $\lambda$ is small or when the insulating boundary is rotating relative to the far fluid. Moreover, they require a prior knowledge of the axial derivative of the axial current at the boundary. Boundary conditions (2.10) are the only physically consistent conditions for the situation under consideration. Precisely the same conditions were used by Benton & Loper (1969) in their study of the linearized transient time development of Ekman–Hartmann layers. They were, however, confronted with a non-uniform approach to the ultimate state since their steady-state solution did not satisfy all the boundary conditions. This is apparently due to the inadequacy of the linearized system, which fails to account for the balance between magnetic diffusion and magnetic convection and for the associated singular terms (in the limit $\epsilon \to 0$).

In order to expose the effect of differential rotation over the whole of the flow regime we employ the method of matched asymptotic expansions. This method is effective, though complicated, largely because the hydromagnetic flow has a distinct double-layered structure. The inner layer is the well-known Ekman–Hartmann layer and results from the viscous–centrifugal–magnetic force balance near the rigid boundary. The outer layer, on the other hand, serves to balance the magnetic diffusion and the magnetic convection of the radial and azimuthal field by the axial inflow. In the nonlinear treatment the outer layer also provides the necessary balance between the electromagnetic body force and the centrifugal force in the far field. In the next two sections we derive asymptotic expansions valid in each of these regions. In this paper we are concerned with the solution for $0 < \epsilon < 1$.

## 3. Asymptotic expansions for the inner layers

For $\epsilon < 1$, it is appropriate to develop each of the unknown functions $F, G, M$ and $N$ as an asymptotic power series in $\epsilon$. The imposed tangential shear (of order $\epsilon$) due to the differential rotation distorts the magnetic field lines, which would otherwise be in the axial direction. This results in the generation of electric currents which persist outside the inner layer. Also the induced inflow at infinity stretches the vortex lines and the magnetic field lines. The magnetic diffusion away from the interface and the magnetic convection towards the interface are balanced to form the outer boundary layer. The nonlinear changes in the applied axial magnetic field are communicated to the insulator through the Ekman–Hartmann layer. We anticipate that such changes are of order unity, so that we assume the field functions to be of the form

$$\left.\begin{aligned} F &= \epsilon F_1 + \epsilon^2 F_2 + \ldots, & G &= \epsilon G_1 + \epsilon^2 G_2 + \ldots, \\ M &= \epsilon M_1 + \epsilon^2 M_2 + \ldots, & N &= N_0 + \epsilon N_1 + \epsilon^2 N_2 + \ldots. \end{aligned}\right\} \tag{3.1}$$

Substituting (3.1) in (2.4)–(2.7) and equating the coefficients of like powers of $\epsilon$, we get

$$N_0 = \text{constant}, \tag{3.2}$$

$$P_1'' - 2iP_1 - 2\lambda(1+2\sigma N_0)Q_1' = 0, \quad Q_1'' - (1+2\sigma N_0)P_1' = 0, \tag{3.3}$$

$$P_2'' - 2iP_2 - 2\lambda(1+2\sigma N_0)Q_2' = P_1^2 - 2F_1 P_1' - 2\lambda\sigma(Q_1^2 - 2N_1 Q_1'), \tag{3.4a}$$

$$Q_2'' - (1+2\sigma N_0)P_2' = 2\sigma(N_1 P_1' - F_1 Q_1'), \tag{3.4b}$$

and so on. In these equations

$$P_j = F_j' + iG_j, \quad Q_j = N_j' + iM_j \quad (j = 1, 2, \ldots). \tag{3.5}$$

The boundary conditions for the inner layer are

$$P_1(0) = i, \quad P_{j+1}(0) = 0, \quad Q_j(0) = 0 = F_j(0) \quad (j = 1, 2, \ldots). \tag{3.6}$$

To complete the specification of the physical system, more conditions on $P_j$, $Q_j$ and $N_j$ are required. These are supplied by matching the inner-layer expansions to the outer expansions.

The solution of the order-$\epsilon^2$ system (3.2)–(3.6) is

$$P_1 = ie^{-mz}, \quad F_1 = \frac{i}{2}\left[\frac{1}{m}(1-e^{-mz}) - \frac{1}{m^*}(1-e^{-m^*z})\right], \tag{3.7a}$$

$$Q_1 = \frac{ik}{m}(1-e^{-mz}), \quad N_1 = \frac{ik}{2}\left[\frac{1}{m}\left(z+\frac{1}{m}e^{-mz}\right) - \frac{1}{m^*}\left(z+\frac{1}{m^*}e^{-m^*z}\right)\right] + A, \tag{3.7b}$$

$$\begin{aligned} P_2 &= \frac{m(mm^* + m^{*2} - 4\lambda\sigma k^2)}{m^{*3}(m+m^*)(2m+m^*)}\{e^{-mz} - \exp[-(m+m^*)z]\} + B(1-e^{-mz}) \\ &+ \frac{z}{2m}\left(1 - \frac{m}{m^*} + \frac{10\lambda\sigma k^2}{m^2} - \frac{6\lambda\sigma k^2}{mm^*} - 8i\lambda\sigma kA\right)e^{-mz} + \frac{\lambda\sigma k^2(m^*-m)z^2}{m^2 m^*}e^{-mz}, \end{aligned} \tag{3.8a}$$

$$
\begin{aligned}
Q_2 = {} & \left(\frac{B}{m}-\frac{m^{*2}+mm^*-4\lambda\sigma k^2}{m^{*3}(m+m^*)(2m+m^*)}\right)(e^{-mz}-1) \\
& +\frac{m(m^{*2}+mm^*-4\lambda\sigma k^2)}{m^{*3}(m+m^*)^2(2m+m^*)}\{\exp[-(m+m^*)z]-1\}+\frac{\lambda\sigma k^2(m-m^*)z^2}{m^3m^*}e^{-mz} \\
& +\left(1-\frac{m}{m^*}+\frac{14\lambda\sigma k^2}{m^2}-\frac{10\lambda\sigma k^2}{mm^*}-8i\lambda\sigma kA\right)\left(\frac{1-e^{-mz}}{2m^3}-\frac{z}{2m^2}e^{-mz}\right) \\
& +\sigma\left[\frac{3(m-m^*)k+2iAm^2m^*}{m^3m^*}(1-e^{-mz})-\frac{(m-m^*)kz}{m^2m^*}e^{-mz}\right. \\
& \left.+\frac{(m-m^*)k}{m^{*2}(m+m^*)^2}\{1-\exp[-(m+m^*)z]\}\right]+Cz,
\end{aligned} \tag{3.8b}
$$

where
$$k = 1+2\sigma N_0, \quad m = (2i+2\lambda k^2)^{\frac{1}{2}}, \tag{3.9}$$

and an asterisk denotes the complex conjugate of the function under it. Expressions for $F_2$ and $N_2$ are obtained after integrating (3.8$a$) and (3.8$b$). The above solution contains four unknown constants, namely $k$ and $A$ (real) and $B$ and $C$ (complex). One more unknown constant will occur in the expression for $N_2$. All these constants are to be evaluated from the conditions for matching (as $z\to\infty$) with the outer expansion.

## 4. Asymptotic expansions for the outer layers

We note from (3.3) that the expansion procedure adopted has reduced in importance the terms representing convection of magnetic field lines. The solutions (3.2), (3.7) and (3.8) may thus only be considered as inner solutions and there must be an outer region in which there is a balance between magnetic diffusion and magnetic convection. We must therefore develop a solution in the outer region which is complementary to, and matches with, the inner solution. In this outer region, which is $O(\epsilon^{-1})$ times as thick as the inner boundary layer, we set

$$\xi = \epsilon z, \quad F(z) = \epsilon f(\xi), \quad G(z) = \epsilon^2 g(\xi), \quad M = \epsilon m(\xi), \quad N(z) = n(\xi). \tag{4.1}$$

Writing $p = f'+ig$ and $q = n'+im$, where a prime now denotes differentiation with respect to $\xi$, we seek a solution of the form

$$p(\xi,\epsilon) = \sum_{j=0}\epsilon^j p_j(\xi), \text{ etc.}$$

of the equations

$$2ip+2\lambda q' = 2\lambda\sigma(q^2-2nq')+\epsilon^2(p''+2fp'-p^2), \tag{4.2a}$$

$$q''-p' = 2\sigma(np'-fq'). \tag{4.2b}$$

under the conditions

$$p(\infty) = 0, \quad q(\infty) = 0, \quad n(\infty) = 0 \tag{4.3}$$

together with the conditions that as $\xi\to 0$ the solution of (4.2) should match the limit as $z\to\infty$ of the solutions (3.2), (3.7) and (3.8). The first two terms of the outer asymptotic expansion are given by

$$ip_0+\lambda q_0' = \lambda\sigma(q_0^2-2n_0q_0'), \quad q_0''-p_0' = 2\sigma(n_0p_0'-f_0q_0'), \tag{4.4}$$

$$ip_1+\lambda q_1' = 2\lambda\sigma(q_0q_1-n_0q_1'-n_1q_0'), \quad q_1''-p_1' = 2\sigma(n_0p_1'+n_1p_0'-f_0q_1'-f_1q_0'). \tag{4.5}$$

It is evident from (4.2)–(4.5) that the induced electromagnetic body force dominates the nonlinear part of the inertia in the outer region. In addition to providing the necessary balance between magnetic diffusion and magnetic convection, the outer region, to order $\epsilon^2$, serves to balance the electromagnetic body force and the centrifugal force. The solution of the nonlinear coupled equations (4.4) is basic to the whole of the above system of differential equations.

Precisely the same equations as (4.4) have been solved numerically by Loper (1972) under different boundary conditions. A significant difference in Loper's analysis is that he introduces a prescribed axial current at infinity. In the present notation the existence of Loper's boundary layer appears to depend upon the sign of the expression

$$f(\infty) - H_0 m(\infty).$$

This expression happens to change sign as the strength of the applied magnetic field is increased. Thus Loper concludes that the solution to his problem does not exist for sufficiently large $\lambda$. In the present case, we maintain $m(\infty) = 0$, and the existence of the boundary layer depends upon the sign of the axial velocity alone. Using the proper matching conditions, which again differ from the boundary conditions (2.11) of Loper, we find that the axial velocity is negative for all values of $\lambda$. Evidently a negative axial velocity always helps to maintain the boundary layers. In the outer region, viscosity manifests itself through the matching conditions.

We proceed to solve (4.4) for $p_0$ and $q_0$ by a method due to Fettis (1955). This method has been used with advantage by Benton (1966, 1973), Stuart (1966) and Chawla (1973) to solve different problems in hydrodynamics and hydromagnetics. The main merit of this method is that it gives the correct, exponential behaviour at infinity.

We write

$$\left.\begin{aligned} p_0 &= sp_{01} + s^2 p_{02} + \dots, \quad q_0 = sq_{01} + s^2 q_{02} + \dots, \\ f_0 &= \alpha + sf_{01} + s^2 f_{02} + \dots, \quad n_0 = sn_{01} + s^2 n_{02} + \dots, \end{aligned}\right\} \tag{4.6}$$

with $\quad q_{0j+1}(0) = 0 = p_{0j}(\infty) = f_{0j}(\infty) = q_{0j}(\infty) = n_{0j}(\infty) \quad (j = 1, 2, \dots),$ (4.7)

where $\alpha$ is a constant to be determined and the parameter $s$ is later set equal to unity. The constant $\alpha$, which is real, gives the axial inflow at infinity and is obtained, along with other constants of integration, from the matching conditions.

Substituting (4.6) in (4.4) and equating the coefficients of equal powers of $s$, we get the following system of differential equations:

$$ip_{01} + \lambda q'_{01} = 0, \quad q''_{01} - p'_{01} + 2\alpha\sigma q'_{01} = 0, \tag{4.8}$$

$$ip_{02} + \lambda q'_{02} = \lambda\sigma [q_{01}^2 - 2n_{01} q'_{01}], \tag{4.9a}$$

$$q''_{02} - p'_{02} + 2\alpha\sigma q'_{02} = 2\sigma [n_{01} p'_{01} - f_{01} q'_{01}], \tag{4.9b}$$

and so on. The solution of (4.8) is

$$p_{01} = -i\lambda\beta l e^{-l\xi}, \quad q_{01} = \beta e^{-l\xi}, \tag{4.10}$$

where $\beta$ is a constant and

$$l = 2\alpha\sigma(1 + i\lambda)/(1 + \lambda^2). \tag{4.11}$$

Using (4.10), the solution of (4.9) is found to be

$$p_{02} = \frac{\lambda\beta\beta^* l}{\alpha l^*}\left\{\left(i\alpha\sigma - \frac{\lambda l(l+l^*)}{l^*}\right)\exp\left[-(l+l^*)\xi\right] + \frac{\lambda l^2}{l^*}e^{-l\xi}\right\}, \tag{4.12a}$$

$$q_{02} = \frac{i\lambda\beta\beta^* l^2}{\alpha l^{*2}}\left(e^{-l\xi} - \exp\left[-(l+l^*)\xi\right]\right). \tag{4.12b}$$

The functions $p_{03}$, $q_{03}$ etc. may be calculated in a similar way. The first four terms in the series (4.6) were calculated for the purpose of matching the inner and outer solutions.

In order to solve the differential set (4.5), again by Fettis's (1955) method, we write

$$\left.\begin{aligned} p_1 &= \delta(2p_0 + \xi p_0') + sp_{11} + s^2 p_{12} + \dots, \\ q_1 &= \delta(q_0 + \xi q_0') + sq_{11} + s^2 q_{12} + \dots, \end{aligned}\right\} \tag{4.13}$$

with $\quad q_{1j+1}^{(0)} = 0 = p_{1j}(\infty) = f_{1j}(\infty) = q_{1j}(\infty) = n_{1j}(\infty) \quad (j = 1, 2, \dots),$ (4.14)

where $\delta$ is a real constant; $\alpha\delta$ gives the axial inflow, of order $\epsilon^2$, at infinity. We substitute (4.6) and (4.13) into the set (4.5) and equate the coefficients of equal powers of $s$. The solutions of the resulting differential equations for $(p_{11}, q_{11})$, $(p_{12}, q_{12})$, etc. are

$$p_{11} = -i\lambda\gamma l\, e^{-l\xi}, \quad q_{11} = \gamma\, e^{-l\xi}, \tag{4.15}$$

$$p_{12} = \frac{\lambda(\beta\gamma^* + \gamma\beta^*)l}{\alpha l^*}\left[\left(i\alpha\sigma - \frac{\lambda l(l+l^*)}{l^*}\right)\exp\left[-(l+l^*)\xi\right] + \frac{\lambda l^2}{l^*}e^{-l\xi}\right], \tag{4.16a}$$

$$q_{12} = \frac{i\lambda(\beta\gamma^* + \gamma\beta^*)l^2}{\alpha l^{*2}}\{e^{-l\xi} - \exp\left[-(l+l^*)\xi\right]\}, \tag{4.16b}$$

and so on, where $\gamma$ is a constant.

The outer solution, as obtained above, contains four unknown constants, namely $\alpha$ and $\delta$ (real) and $\beta$ and $\gamma$ (complex). These are to be determined from the matching conditions.

## 5. Matching of the inner and outer expansions

We now evaluate the constants $k$, $A$, $B$ and $C$ and $\alpha$, $\delta$, $\gamma$ and $\beta$ occurring in the inner and outer expansions respectively. These are obtained from the condition that the inner solution (as $z \to \infty$) must match properly with the outer one (as $\xi \to 0$). For a proper matching the velocity and magnetic fields and also the current density must be continuous across the two regions of fluid flow. This means that the inner and outer expansions of the functions $P$, $F$, $Q$, $N$ and $Q'$ must match to the same order in $\epsilon$.

The matching condition $Q_1(\infty) = q_0(0)$ immediately leads to

$$ik/m = \beta = ik(2i + 2\lambda k^2)^{-\frac{1}{2}}. \tag{5.1}$$

The real constants $k$ and $\alpha$ are then given by the conditions $F_1(\infty) = f_0(0)$ and $N_0(\infty) = n_0(0)$. Using a four-term series solution (4.6) for the outer expansion (after setting $s = 1$), these conditions lead to the following equations for $\alpha$ and $k$:

$$
\begin{aligned}
(i/m - i/m^*)\,\alpha^3 = {} & 2\alpha^4 + i\lambda(\beta-\beta^*)\alpha^3 + \lambda^2\beta\beta^*\alpha^2 \\
& + \frac{\lambda^2\beta\beta^*}{\lambda^2+9}\,[(\lambda^2-3)(\beta+\beta^*) + 4i\lambda(\beta-\beta^*)]\,\alpha \\
& + \lambda^2\beta\beta^*\left[\frac{3+22\lambda^2-5\lambda^4}{4(9+\lambda^2)}\,\beta\beta^* + \frac{6-41\lambda^2+25\lambda^4}{(4+\lambda^2)(9+\lambda^2)}(\beta^2+\beta^{*2})\right. \\
& \left. - \frac{i\lambda(17-51\lambda^2+4\lambda^4)}{(4+\lambda^2)(9+\lambda^2)}(\beta^2-\beta^{*2})\right],
\end{aligned} \tag{5.2a}
$$

$$
\begin{aligned}
(k-1)\alpha^4 = {} & \tfrac{1}{2}\,[i\lambda(\beta-\beta^*) - (\beta+\beta^*)]\,\alpha^3 + \lambda^2\beta\beta^*\alpha^2 \\
& + \frac{\lambda^2\beta\beta^*}{4(\lambda^2+9)}\,[(9\lambda^2-15)(\beta+\beta^*) - i\lambda(\lambda^2-23)(\beta-\beta^*)]\,\alpha \\
& + \lambda^2\beta\beta^*\left[\frac{3+11\lambda^2-2\lambda^4}{2(9+\lambda^2)}\,\beta\beta^* + \frac{12-81\lambda^2+50\lambda^4-\lambda}{4(4+\lambda^2)(9+\lambda^2)}(\beta^2+\beta^{*2})\right. \\
& \left. - \frac{i\lambda(10-23\lambda^2+3\lambda^4)}{(4+\lambda^2)(9+\lambda^2)}(\beta^2-\beta^{*2})\right].
\end{aligned} \tag{5.2b}
$$

Although not containing $\sigma$ explicitly, equations (5.2), together with (5.1), are highly complicated to solve. We solve these equations for small and large values of the dimensionless parameter $\lambda$ and obtain the following expressions for $\alpha$, $k$ and $\beta$:

$$
\alpha = \begin{cases} \dfrac{1}{2}\left(1 + \dfrac{3\lambda}{8} - \dfrac{77\lambda^2}{384} + O(\lambda^3)\right) & (\lambda \text{ small}), \quad (5.3a) \\ (\tfrac{1}{2}\lambda)^{\frac{1}{2}}\,(1 + O(\lambda^{-2})) & (\lambda \text{ large}), \quad (5.3b) \end{cases}
$$

$$
k = \begin{cases} \dfrac{1}{2}\left(1 - \dfrac{\lambda}{4} + \dfrac{59\lambda^2}{192} + O(\lambda^3)\right) & (\lambda \text{ small}), \quad (5.4a) \\ 1 + O(\lambda^{-2}) & (\lambda \text{ large}), \quad (5.4b) \end{cases}
$$

$$
\beta = \begin{cases} \dfrac{1}{4}\left(1 - \dfrac{3\lambda}{8} + \dfrac{17\lambda^2}{24} + O(\lambda^3)\right) + \dfrac{i}{4}\left(1 - \dfrac{\lambda}{8} + \dfrac{25\lambda^2}{48} + O(\lambda^3)\right) & (\lambda \text{ small}), \quad (5.5a) \\ \left(\dfrac{1}{2\lambda}\right)^{\frac{1}{2}}\left(\dfrac{1}{2\lambda} + i + O(\lambda^{-2})\right) & (\lambda \text{ large}). \quad (5.5b) \end{cases}
$$

We now obtain $B$ from the condition $P_2(\infty) = p_0(0)$:

$$
B = \begin{cases} \tfrac{1}{4}\sigma\lambda(1 - \tfrac{1}{6}\lambda) - \tfrac{1}{8}i\sigma\lambda(1 - \tfrac{3}{2}\lambda) + O(\lambda^3) & (\lambda \text{ small}), \quad (5.6a) \\ 2i\sigma(1 - 3i/\lambda) + O(\lambda^{-2}) & (\lambda \text{ large}). \quad (5.6b) \end{cases}
$$

Once $B$ has been determined, the constant $C$ in the expression (3.8$b$) for $Q_2$ is obtained from the condition that the tangential electric currents across the two regions must be continuous; that is $Q_2'(\infty) = q_0'(0)$. We get

$$
C = -\sigma k/m^2 - iB/\lambda k. \tag{5.7}
$$

We note that, in the above process of matching the inner and outer solutions, proper account has already been taken of the non-exponential coefficients of $z$

in $Q_2$, $F_2$ and $N_1$, through the matching of $Q_2'$, $P_2$ and $Q_1$, respectively. We therefore set

$$\left.\begin{aligned}[N_1-\tfrac{1}{2}(\beta+\beta^*)z]_{z\to\infty} &= n_1(0),\\ [Q_2-Cz]_{z\to\infty} = q_1(0), \quad [F_2-\tfrac{1}{2}(B+B^*)z]_{z\to\infty} &= f_1(0),\end{aligned}\right\} \tag{5.8}$$

in order to evaluate the remaining constants $A$, $\delta$ and $\gamma$. These conditions lead to a complicated coupled system of equations for $A$, $\delta$ and $\gamma$. We omit details but give the values of these constants for small and large values of $\lambda$:

$$A = \begin{cases} \dfrac{3}{80\sigma}-\dfrac{5}{16}-\dfrac{129\lambda}{3200\sigma}+\dfrac{137\lambda}{640}+O(\lambda^2) & (\lambda \text{ small}), & (5.9a)\\ -2+O(\lambda^{-2}) & (\lambda \text{ large}), & (5.9b)\end{cases}$$

$$\delta = \begin{cases} -\dfrac{3}{10}+\dfrac{9\lambda}{500}-\dfrac{47\lambda\sigma}{80}+O(\lambda^2) & (\lambda \text{ small}), & (5.10a)\\ O(\lambda^{-\frac{3}{2}}) & (\lambda \text{ large}), & (5.10b)\end{cases}$$

$$\gamma = \begin{cases} -\dfrac{3}{8}+\dfrac{5\sigma}{16}-\dfrac{69\lambda}{800}+\dfrac{69\lambda\sigma}{320}+i\left(\dfrac{9}{80}-\dfrac{5\sigma}{16}+\dfrac{3\lambda}{100}+\dfrac{9\lambda\sigma}{80}\right)+O(\lambda^2) & (\lambda \text{ small}), & (5.11a)\\ 4i\sigma/(2\lambda)^{\frac{1}{2}}+O(\lambda^{-\frac{3}{2}}) & (\lambda \text{ large}). & (5.11b)\end{cases}$$

## 6. Discussion of results

Although the asymptotic expansions for the flow and magnetic field functions $F$, $G$, $M$ and $N$ constitute an essentially nonlinear multilayered structure, they vary on two vastly different spatial scales. The two regions governed by the two length scales are such that the thickness of the inner layer decreases and that of the outer layer increases as the strength of the applied magnetic field is increased. Following Benton & Loper (1969), we call the two regions the Ekman–Hartmann layers (EHL) and the magnetic diffusion region (MDR). The edge of the EHL provides a smooth transition between the two regions, and the matching conditions for the inner and outer asymptotic solutions were chosen accordingly.

Actually there are four regions of influence of the nonlinear hydromagnetic interaction due to the differential rotation: the interior of the insulator, the EHL, the MDR and the region far away from the interface (outside the MDR). The present solution provides all the information about the physical characteristics in each of these regions as our focus of attention moves from the rigid body to the far field. We now proceed to consider some important features of the flow and magnetic field in different regions by making use of the simplifying calculations of the previous section.

### 6.1. *Ekman–Hartmann layers*

The EHL result from the viscous–centrifugal–magnetic force balance near the rigid boundary. The thickness of the EHL is $O(m_r^{-1})$, where

$$m = m_r+im_i = [(1+\lambda^2k^4)^{\frac{1}{2}}+\lambda k^2]^{\frac{1}{2}}+i\,[(1+\lambda^2k^4)^{\frac{1}{2}}-\lambda k^2]^{\frac{1}{2}}, \tag{6.1}$$

and $k$ varies from $\frac{1}{2}$ to 1 as the strength of the applied magnetic field is increased. In the linear theory (Benton & Loper 1969) $k = 1$. Although the thickness of the

EHL decreases with increasing $\lambda$, the EHL are thicker (for a given strength of the applied magnetic field) than is predicted by the linear theory.

The velocity field $P = \epsilon P_1 + \epsilon^2 P_2$, given by (3.7) and (3.8), reduces uniformly to the result previously obtained by Rogers & Lance (1960) for the case $\lambda = 0$. However, it differs significantly from the corresponding result of Benton & Chow (1972). The EHL, like Ekman layers, possess oscillatory structure, but the applied magnetic field tends to suppress their oscillatory character.

The real importance of the EHL rests in their ability to induce mass flux (suction) and electric currents into the MDR. The Ekman–Hartmann pumping into the outer layer is given by

$$\epsilon f(0) = \begin{cases} \dfrac{\epsilon}{2}\left[\left(1-\dfrac{\lambda}{8}+\dfrac{5\lambda^2}{128}\right)-\dfrac{3\epsilon}{10}\left(1+\dfrac{\lambda}{20}-\dfrac{\sigma\lambda}{8}\right)\right] & (\lambda \text{ small}), \quad (6.2a) \\ O(\epsilon\lambda^{-\frac{3}{2}}) & (\lambda \text{ large}), \quad (6.2b) \end{cases}$$

whereas the Hartmann axial current induced into the outer layer is given by

$$\epsilon m(0) = \begin{cases} \dfrac{\epsilon}{4}\left(1-\dfrac{\lambda}{8}+\dfrac{25\lambda^2}{48}\right)-\dfrac{\epsilon^2}{4}\left(\dfrac{3}{4}+\dfrac{5\sigma}{4}-\dfrac{9\lambda}{25}+\dfrac{19\sigma\lambda}{10}\right) & (\lambda \text{ small}), \quad (6.3a) \\ \epsilon(1+4\epsilon\sigma)/(2\lambda)^{\frac{1}{2}}+O(\lambda^{-\frac{3}{2}}) & (\lambda \text{ large}). \quad (6.3b) \end{cases}$$

Equation (6.2$a$) reduces to the result of Rogers & Lance (1960) for $\lambda = 0$. Compared with the linear Ekman–Hartmann pumping, which is equal to

$$[(1+\lambda^2)^{\frac{1}{2}}-\lambda]^{\frac{1}{2}}/2(1+\lambda^2)^{\frac{1}{2}},$$

the nonlinear theory gives a larger mass flux in the limit $\epsilon \to 0$; the mass flux itself decreases with increasing $\lambda$. This is evidently due to the fact that the magnetic field supporting the Ekman–Hartmann layer is weaker than the applied magnetic field; the hydromagnetic interaction owing to differential rotation brings about nonlinear changes of order unity in the applied field. We shall return to this point later on. To zeroth order in $\epsilon$, the axial magnetic field ($= kH_0$) remains constant through the thickness of the EHL.

### 6.2. *Magnetic diffusion region*

The development and establishment of the MDR, as the outer layer, are important features of the hydromagnetic interaction in a rotating environment. The MDR primarily results from the balance between the outward magnetic diffusion and the inward convection. It also serves to balance the induced electromagnetic body force and the inertia of induced rotation. The thickness of this region, which is much thicker than the EHL, is $O(\epsilon^{-1}l_r^{-1})$, where

$$l = l_r + il_i = 2\alpha\sigma(1+i\lambda)/(1+\lambda^2)$$
$$= \begin{cases} \sigma\left(1+\dfrac{3\lambda}{8}-\dfrac{461\lambda^2}{384}\right)(1+i\lambda) & (\lambda \text{ small}), \quad (6.4a) \\ \sigma(2/\lambda^3)^{\frac{1}{2}}(1+i\lambda) & (\lambda \text{ large}). \quad (6.4b) \end{cases}$$

From the relationship between the various parameters, we find that the thickness of the outer region in general increases from $O(\eta/(\nu\Omega)^{\frac{1}{2}}\epsilon)$ to $O(A_0^3/\epsilon\Omega^2(\nu\eta)^{\frac{1}{2}})$ with increasing strength of the applied magnetic field; $A_0$ is the Alfvén velocity $(\mu H_0^2/\rho)^{\frac{1}{2}}$. For finite values of $\epsilon$ and $\sigma$, the outer region extends to finite distances. In contrast, the analyses of Gilman & Benton (1968) and Benton & Chow (1972) imply that the outer layer (MDR) extends to spatial infinity. Clearly the MDR is also nonlinear and is oscillatory in character. The spatial oscillations become more pronounced with increasing $\lambda$. For all finite, though small, values of the magnetic Prandtl number $\sigma$, the diffusivity of the fluid is important in determining the conditions in the outer region.

The azimuthal velocity of the edge of the EHL is given by

$$\epsilon^2 g(0) = \begin{cases} -\frac{1}{8}\sigma\lambda\,(1-\frac{3}{2}\lambda)\epsilon^2 & (\lambda \text{ small}), \quad (6.5a) \\ 2\sigma\epsilon^2 + O(\lambda^{-2}\epsilon^2) & (\lambda \text{ large}), \quad (6.5b) \end{cases}$$

whereas the radial velocity is given by

$$\epsilon^2 f'(0) = \begin{cases} \frac{1}{4}\sigma\lambda(1-\frac{1}{6}\lambda)\epsilon^2 & (\lambda \text{ small}), \quad (6.6a) \\ 6\sigma\epsilon^2/\lambda + O(\lambda^{-3}\epsilon^2) & (\lambda \text{ large}). \quad (6.6b) \end{cases}$$

For low values of $\lambda$, the region just outside the EHL develops a rotation, albeit very slow, in the direction opposite to that of the main rigid-body rotation. For large values of $\lambda$, on the other hand, the induced tangential electromagnetic force makes this region rotate in the same direction as the basic rotation. This, together with the fact that the thickness of the MDR increases with $\lambda$, brings additional layers of the conducting fluid under the influence of the differential rotation of the rigid boundary as the strength of the applied magnetic field is increased. As a result, for sufficiently large values of $\lambda$, the transitional region of the fluid will rotate with angular velocity $\Omega(1+2\sigma\epsilon^2)$.

An important consequence of the hydromagnetic interaction under consideration is the distortion of the applied axial magnetic field owing to the stretching of the vortex lines by the Ekman–Hartmann pump. In contrast to the basic applied magnetic field $H_0$, the net external axial field supporting the EHL is given by

$$H_0(1+2\sigma n(0)) = H_0(k+2\sigma A\epsilon)$$

$$= \begin{cases} H_0\left[\frac{1}{2}\left(1-{}^{\lambda}+\frac{59\lambda}{192}\right)+\epsilon\left(\frac{3}{40}-\frac{5\sigma}{8}-\frac{129\lambda}{1600}+\frac{137\sigma\lambda}{320}\right)\right] & (\lambda \text{ small}), \quad (6.7a) \\ H_0(1-4\sigma\epsilon)+O(\lambda^{-2}) & (\lambda \text{ large}). \quad (6.7b) \end{cases}$$

Compared with this, the normal field induced within the insulating boundary is

$$H_0(1+2\sigma N(0)) = H_0(1+2\sigma N_0(0)+2\epsilon\sigma N_1(0))$$

$$= \begin{cases} \frac{H_0}{2}\left(1-\frac{\lambda}{4}+\frac{59\lambda^2}{192}\right)+\epsilon H_0\left(\frac{3}{40}-\frac{\sigma}{8}-\frac{129\lambda}{1600}+\frac{97\sigma\lambda}{320}\right) & (\lambda \text{ small}), \quad (6.8a) \\ H_0(1-4\sigma\epsilon)+O(\lambda^{-2}) & (\lambda \text{ large}). \quad (6.8b) \end{cases}$$

We find that substantial changes (of order unity) in the basic axial field over the entire width of the flow regime can be effected and sustained by even a small differential rotation. It is interesting that the bulk of the distortions of the applied field take place within the MDR, although these changes are brought about by the Ekman–Hartmann pump at the edge of the EHL. Roberts & Scott (1965) have stated that the changes in the magnetic field across the boundary layer at the core–mantle interface within the earth must be small. Evidently their argument involving a non-rotating boundary is not valid for a rotating boundary. The differential rotation across different layers of the EHL stretches the axial field into an azimuthal component [see (6.3)]. The body force associated with this azimuthal component is primarily responsible for the induced angular velocity of the main body of the MDR [see (6.5)].

Associated also with the azimuthal component is an axial current which turns into the radial direction [see (2.9)] within the MDR and interacts with the axial magnetic field to produce an electromagnetic body torque. Fluid is then propelled outwards by centrifugal action [see (6.5)] and, by continuity, an axial inflow is induced at the outer edge of the MDR. The suction velocity in the far field is given by

$$\epsilon f(\infty) = \epsilon\alpha(1+\epsilon\delta)$$
$$= \begin{cases} \dfrac{\epsilon}{2}\left(1+\dfrac{3\lambda}{8}-\dfrac{77\lambda^2}{384}\right)-\dfrac{3\epsilon^2}{20}\left(1+\dfrac{3\lambda}{10}+\dfrac{47\sigma\lambda}{24}\right) & (\lambda \text{ small}), \quad (6.9a) \\ \epsilon(\tfrac{1}{2}\lambda)^{\frac{1}{2}}(1+O(\lambda^{-2})) & (\lambda \text{ large}). \quad (6.9b) \end{cases}$$

For $\lambda = 0$, (6.9$a$) reduces to the result obtained by Rogers & Lance (1960). The inflow velocity increases with $\lambda$. This is natural in view of the fact that the thickness of the MDR increases with $\lambda$, and more and more fluid is required to replace the outward radial flow within the MDR. For $\epsilon \rightarrow 0$, the present nonlinear analysis gives an inflow velocity (for given $\lambda$) smaller than the value

$$(= \tfrac{1}{2}[(1+\lambda^2)^{\frac{1}{2}}+\lambda]^{\frac{1}{2}})$$

obtained by Benton & Loper (1969) from a transient linearized analysis. However, for large values of $\lambda$, (6.9$b$) agrees with the corresponding results of Benton & Loper (1969). Nevertheless, the nonlinearity, which is undoubtedly more pronounced for weaker hydromagnetic coupling, retains its importance (for large $\lambda$) as a balancing factor (even for $\epsilon \rightarrow 0$) between different effective forces in the far field.

The time for the transient evolution (called the spin-up time) of the double-decker boundary layers is an important time scale. Benton & Loper (1969) have shown that with increasing strength of the applied magnetic field the Ekman–Hartmann layer develops more rapidly (see also Chawla 1972). At the same time the induced tangential electromagnetic body force acts to spin up the fluid in the outer region. For a sufficiently strong applied magnetic field the Ekman–Hartmann layers develop in a time of order $2/(\lambda\Omega)$ (Benton & Loper 1969) and during that time the edge of the EHL attains an angular velocity $\Omega(1+2\sigma\epsilon^2)$. In the ultimate state the inflow velocity at the outer edge of the MDR (of thickness $(\epsilon\sigma)^{-1}(\frac{1}{2}\lambda^3)^{\frac{1}{2}})$ is $\epsilon(\frac{1}{2}\lambda)^{\frac{1}{2}}$. Following Greenspan (1968, p. 36), we find that the spin-up time is of order $\lambda/\Omega$ and thus increases with $\lambda$. We infer that in general the spin-up is slowed down by hydromagnetic coupling. In contrast, Benton (1973) finds that the spin-up of a confined electrically conducting fluid is accelerated by a magnetic field. If the fluid has a finite axial depth, it may happen that the thickness of the MDR exceeds the axial extent of the fluid. In such a case the outer layer discussed in this paper would not be perceived.

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